

SOME COMBINATORIAL ASPECTS OF DISCRETE NON-LINEAR POPULATION DYNAMICS

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ABSTRACT. Motivated by issues arising in population dynamics, we consider the problem of iterating a given analytic function a number of times. We use the celebrated technique known as Carleman linearization that turns (for a certain class of functions) this problem into simply taking the power of a real number. We expand this method, showing in particular that it can be used for population models with immigration, and we also apply it to the famous logistic map. We also are able to give a number of results for the invariant density of this map, some being related to the Carleman linearization.

Keywords: non-linear population dynamics without and with immigration, Carleman transfer matrix, logistic map.

1. INTRODUCTION

We consider simple 1-dimensional discrete-time dynamics: $x_{n+1} = \phi(x_n)$, $x_0 = x$, with the evolution mechanism $\phi(\cdot)$ being an analytic function. Our main interest in these problems arises from population dynamics models describing the temporal evolution of some population with size $x_n \geq 0$. We first assume $\phi(0) = 0$ (no immigration). Such non-linear models are amenable to a Carleman linearization giving x_n from the initial condition x in terms of the n -th power of some upper-triangular infinite-dimensional transfer matrix which can be diagonalized. Equivalently, ϕ is h -conjugate to the linear map λx for some Carleman function h , would $\lambda = \phi'(0) \neq \{-1, 0, 1\}$. The coefficients of h , as a power series in x , are obtained from the left eigenvector of P with the eigenvalue λ . The Carleman linearization technique goes back the 60' ([7], [10], [12], [9]). When $\lambda = 1$ (the critical case), we give the linear Carleman representation of x_n , using a 'Jordanization' technique. Special such models arising in population dynamics are defined and investigated. We next consider the problem of computing the invariant density (and its support) of the dynamics in a chaotic population model regime, including quadratic maps. The study of the invariant measures of quadratic and related maps has a very long story starting in the 70' ([3], [16], [2], [11], [4]). We show that in some special cases, the h -conjugate representation of ϕ is useful for that purpose. We illustrate our point of view on the celebrated logistic population model $\phi(x) = rx(1-x)$. Next we consider $\phi_0(x) = c + \phi(x)$, modelling some population dynamics with immigration $c > 0$. In the presence of a fixed point for ϕ_0 , such models are also Carleman linearizable; equivalently, ϕ_0 is shown to be g -conjugate now to an affine map for some explicit Carleman function g . As an illustration, we finally deal with the logistic population model with immigration. We develop its intimate relation to a family of companion logistic population models without immigration, the former

being obtained from the latter through a suitable affine transformation. We exploit this deep connection to determine under which condition the logistic model with immigration is chaotic or not and, using this observation, we compute in some cases its invariant density.

The precise organization of the paper is as follows:

In Section 2, we recall and develop the Carleman transfer matrix linearization technique, including:

- its link with a conjugate representation of the map ϕ in the case $\phi(0) = 0$ and $\phi'(0) \neq \{-1, 0, 1\}$.
- the application of this scheme to the specific critical case $\phi'(0) = 1$, leading to a method akin to the Jordanization of a matrix.
- the consequences of this construction in terms of the invariant measure of the dynamical system.
- the application of this general setting to a class of specific population evolution models.

In Section 3, using the above tools, we focus on the one-parameter logistic population model. Specifically, we characterize the loci and the types of the divergence of its invariant measure and we give a way to compute the disconnected components of its support for some parameter r range. For some values of the parameter r , we show how to compute explicitly the invariant density of the system.

In Section 4, expanding the tools introduced in Section 2 to include maps obeying $\phi_0(0) \neq 0$, we study the effect of adding immigration to population dynamics models, by relating it to an affine conjugate equivalent of the new mechanism with immigration $\phi_0(0) > 0$. Once again, we apply these results to the logistic map. Our main result on this point is summarized in Figure 1 showing the values of the parameters (r, c) for which this topological conjugation is admissible. As a consequence, the chaoticity of the logistic model with immigration is revealed by the one of the corresponding model without immigration.

2. CARLEMAN MATRIX IN THE TRIANGULAR CASE $\alpha_0 = 0$

With α_k , $k \geq 1$, real numbers, let $\phi(x) = \sum_{k \geq 1} \alpha_k x^k$, $\alpha_1 \neq 0$, be some smooth power series defined (convergent) in some neighborhood $x_c^- < x < x_c^+$, of the origin where $-\infty \leq x_c^- < 0 < x_c^+ \leq \infty$ ¹. We avoid the trivial linear case $\phi(x) = \alpha_1 x$. We shall let $I_c = (x_c^-, x_c^+)$ be the interval of convergence. Note $\alpha_0 = 0$. Consider the dynamical system

$$(1) \quad x_{n+1} = \phi(x_n), \quad x_0 = x.$$

Define the infinite-dimensional (Carleman) upper-triangular matrix

$$(2) \quad P(k, k') = \left[x^{k'} \right] \phi(x)^k, \quad k' \geq k \geq 1.$$

¹We also assume that ϕ is absolutely convergent with radius of convergence $0 < r_c \leq \min(-x_c^-, x_c^+)$

By Faà di Bruno formula (see e.g. [5], Tome 1, p. 148), with $\widehat{B}_{k,l}(\alpha_1, \alpha_2, \dots)$ the (ordinary) Bell polynomials in the coefficients $\alpha_k := [x^k] \phi(x)$ of $\phi(x)$,

$$(3) \quad P(k, k') = \widehat{B}_{k',k}(\alpha_1, \alpha_2, \dots, \alpha_{k'-k+1}) = k! \sum_{c_l}^{**} \prod_{l=1}^{k'-k+1} \frac{\alpha_l^{c_l}}{c_l!},$$

where the last double-star summation runs over the integers $c_l \geq 0$ such that $\sum_{l=1}^{k'-k+1} c_l = k$ and $\sum_{l=1}^{k'-k+1} l c_l = k'$ (there are $p_{k,k'}$ terms in this sum, the number of partitions of k' into k summands). In particular $P(k, k) = \alpha_1^k$ and $P(k, k+1) = k \alpha_2 \alpha_1^{k-1}$. P is called the Carleman² matrix of ϕ . If for example, $\phi(x) = x - x^2$, $P(k, k') = \widehat{B}_{k',k}(z, -1, 0, \dots)|_{z=1}$, the Hermite polynomials evaluated at $z = 1$. We conclude (see [10], [15], [1], [12] and [9]):

Proposition: With $\mathbf{e}'_1 = (1, 0, 0, \dots)$ and $\mathbf{x}' = (x, x^2, \dots)^3$,

$$(4) \quad x_n = \mathbf{e}'_1 P^n \mathbf{x}$$

where P is an upper-triangular ‘transfer’ matrix with $P(k, k) = \alpha_1^k =: \lambda^k$, $k \geq 1$ (the eigenvalues of P).

From (1), x_n is also $x_n = \phi_n(x)$ where ϕ_n is the n -th iterate of ϕ by composition and so (4) is an alternative linear representation of x_n . Note

$$\sum_{n \geq 0} \lambda^n x_n = \mathbf{e}'_1 (I - \lambda P)^{-1} \mathbf{x},$$

involving the resolvent of P .

Remark: We have

$$\begin{aligned} \frac{1}{1 - u\phi(x)} &= 1 + \sum_{k \geq 1} u^k \phi(x)^k = 1 + \sum_{k \geq 1} u^k \sum_{k' \geq k} x^{k'} [x^{k'}] \phi(x)^k \\ &= 1 + \sum_{k \geq 1} u^k \sum_{k' \geq k} x^{k'} P(k, k') = 1 + \sum_{k' \geq 1} x^{k'} \sum_{k=1}^{k'} u^k [x^{k'}] \phi(x)^k \\ &= \sum_{k=1}^{k'} u^k P(k, k') = [x^{k'}] \frac{1}{1 - u\phi(x)}. \end{aligned}$$

With $\phi_n(x) = x_n$, the n -th iterate of ϕ and $\mathbf{u}' := (u, u^2, \dots)$

$$\frac{1}{1 - u\phi_n(x)} = 1 + \sum_{k \geq 1} u^k \sum_{k' \geq k} x^{k'} P^n(k, k') = 1 + \sum_{k' \geq 1} x^{k'} \sum_{k=1}^{k'} u^k P^n(k, k') = 1 + \mathbf{u}' P^n \mathbf{x}.$$

Taking the derivative with respect to u at $u = 0$ gives $\phi_n(x) = x_n = \mathbf{e}'_1 P^n \mathbf{x}$. Taking the k -th derivative with respect to u at $u = 0$ gives $x_n^k = \mathbf{e}'_k P^n \mathbf{x}$. We conclude:

²Carleman matrices are easily seen to be the transpose of Bell matrices.

³Throughout, a boldface variable, say \mathbf{x} , will represent a column-vector and its transpose, say \mathbf{x}' , will be a row-vector.

Proposition: If $\psi(x) = \sum_{k \geq 1} \psi_k x^k$ is some smooth observable, defining $\psi' = (\psi_1, \psi_2, \dots)$, therefore

$$(5) \quad \psi(x_n) = \sum_k \psi_k \mathbf{e}'_k P^n \mathbf{x} = \psi' P^n \mathbf{x}.$$

This generalizes (4).

By Cauchy formula, whenever ϕ is defined on the unit circle, we also have the Fourier representation

$$P(k, k') = \frac{1}{2\pi} \int_0^{2\pi} e^{ik'\theta} \phi(e^{-i\theta})^k d\theta.$$

Chaos for (1) is sometimes characterized by the positivity of its Lyapounov exponent defined by

$$\lambda(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log |\phi'(x_n)|,$$

for almost all x . Considering the sensitivity to the initial condition problem, we have $J_{n+1} := dx_{n+1}/dx = \phi'(x_n) dx_n/dx = \phi'(x_n) J_n$. Therefore $|J_N| = \left(\prod_{n=0}^{N-1} |\phi'(x_n)| \right)$ and $\lambda_N(x) := \frac{1}{N} \log |J_N| \rightarrow \lambda(x)$. Letting $\mathbf{x}' = (x, x^2, \dots)$, $\bar{\mathbf{x}} := (1, x, x^2, \dots)$ and $D := \text{diag}(1, 2, 3, \dots)$ so that $d\mathbf{x}/dx = D\bar{\mathbf{x}}$, we observe

$$J_N := dx_N/dx = \mathbf{e}'_1 P^N D\bar{\mathbf{x}}.$$

2.1. The case $|\lambda| \neq 1$. Suppose $\lambda := \alpha_1 \neq \pm 1$ and let $\mathbf{v}'_k P = \lambda^k \mathbf{v}'_k$, define the left-row-eigenvector \mathbf{v}'_k of P associated to the eigenvalue $\lambda_k := \lambda^k$. Then, with $k' > k \geq 1$,

$$\mathbf{v}_k(k') = \left(\lambda^k - \lambda^{k'} \right)^{-1} \sum_{l=1}^{k'-1} P(l, k') \mathbf{v}_k(l)$$

gives the entries $\mathbf{v}_k(k')$, $k' > k \geq 1$, of \mathbf{v}'_k by recurrence; and $\mathbf{v}_k(k)$ can be left undetermined. Developing, for $k' > k$, we get

Proposition: Suppose $\lambda = \alpha_1 \neq 1$, then

$$\begin{aligned} \mathbf{v}_k(k') &= \mathbf{v}_k(k) \sum_{j=2}^{k'-k+1} \sum_{d_1+\dots+d_{j-1}=k'-k}^* \prod_{l=1}^{j-1} \frac{P(d_1+\dots+d_{l-1}+k, d_1+\dots+d_l+k)}{\lambda^k - \lambda^{d_1+\dots+d_l+k}}, \text{ or} \\ \mathbf{v}_k(k') &= \mathbf{v}_k(k) \sum_{j=2}^{k'-k+1} \sum_{k=k_1 < k_2 < \dots < k_{j-1} < k_j=k'} \prod_{l=1}^{j-1} \frac{P(k_l, k_{l+1})}{\lambda^k - \lambda^{k_{l+1}}}, \end{aligned}$$

where the star-sum in the first identity (involving the integer partition of $k' - k$) runs over the integers $d_l \geq 1$ summing to $k' - k$.

Thus, with $V = [\mathbf{v}_1, \dots, \mathbf{v}_k, \dots]'$, $VP = D_\lambda V$ where V is upper-triangular (so here invertible) and $D_\lambda = \text{diag}(\lambda, \lambda^2, \dots)$,

$$(6) \quad x_n = \mathbf{e}'_1 V^{-1} D_\lambda^n V \mathbf{x}.$$

This shows that a general non-linear dynamical system (1) generated by ϕ with $\phi(0) = \alpha_0 = 0$ and $\alpha_1 \neq \{-1, 0, 1\}$ is in fact a linear infinite-dimensional system with ‘transfer matrix’ P which can effectively be diagonalized.

Proposition: Defining $h_k(x) = \mathbf{v}'_k \mathbf{x} = \sum_{k' \geq k} \mathbf{v}_k(k') x^{k'}$, this also means that for all $k \geq 1$

$$x_n = h_k^{-1} \left(\lambda^{nk} h_k(x) \right).$$

Proof: Indeed, $(P\mathbf{x})_l = \phi(x)^l$ and $\mathbf{v}'_k P = \lambda^k \mathbf{v}'_k \Rightarrow \mathbf{v}'_k P \mathbf{x} = \lambda^k \mathbf{v}'_k \mathbf{x} \Rightarrow h_k(\phi(x)) = \lambda^k h_k(x)$ where $h_k(x) = \mathbf{v}'_k \mathbf{x} = \sum_{k' \geq k} \mathbf{v}_k(k') x^{k'}$, for all $k \geq 1$. Iterating, $x_1 = \phi(x) = h_k^{-1} \left(\lambda^k h_k(x) \right) \Rightarrow x_2 = \phi(x_1) = h_k^{-1} \left(\lambda^{2k} h_k(x) \right) \Rightarrow x_n = \phi(x_{n-1}) = h_k^{-1} \left(\lambda^{nk} h_k(x) \right)$.

When $k = 1$, with $\mathbf{v} := \mathbf{v}_1$ and

$$(7) \quad h(x) = \mathbf{v}' \mathbf{x} = \sum_{k' \geq 1} \mathbf{v}(k') x^{k'} =: \sum_{k' \geq 1} h_{k'} x^{k'}$$

and h^{-1} the analytic continuation of the inverse function of h , both obeying $h(0) = h^{-1}(0) = 0$, this in particular means

$$(8) \quad x_1 := \phi(x) = h^{-1}(\lambda h(x)) \text{ and } x_n = h^{-1}(\lambda^n h(x)).$$

The importance of this formula relies on the fact that, in this way, x_n can be evaluated directly, for any initial point x , without actually computing the intermediate values x_1, \dots, x_{n-1} . But this is at the expense of the computation of h and h^{-1} , which are simple special functions only in some exceptional situations.

Remarks:

(i) In a neighborhood of the origin, the analytic continuation of the inverse function and the inverse function itself defined on the range of h indeed coincide. By Lagrange inversion formula, h^{-1} , viewed as a power series, is locally defined in some neighborhood of the origin⁴, with coefficients

$$g_k := [x^k] h^{-1}(x) = \frac{1}{k} [x^{k-1}] (h(x)/x)^{-k}.$$

By Faà di Bruno formula, $g_1 = (h^{-1})'(0) = 1/h_1$ and for $k \geq 2$

$$(9) \quad g_k = h_1^{-k} \sum_{l=1}^{k-1} (-1)^l (k+l-1)_{k-1} \widehat{B}_{k-1,l}(\widehat{h}_1, \widehat{h}_2, \dots),$$

where $(k)_l = k(k-1)\dots(k-l+1)$, $\widehat{h}_k = h_{k+1}/h_1$ and $\widehat{B}_{k,l}$ the ‘ordinary’ Bell polynomials.

(ii) If $0 < \lambda < 1$ (a subcritical case), (8) says $x_n \rightarrow 0$ whatever x as $n \rightarrow \infty$. Furthermore, a first order Taylor development shows that $x_n = h^{-1}(0) + (h^{-1})'(0) \lambda^n h(x)$ with $h^{-1}(0) = 0$ and $(h^{-1})'(0) = 1/h_1$. So x_n goes to 0 geometrically fast at rate λ and $\lambda^{-n} x_n \rightarrow h(x)/h_1$.

⁴If h has a positive convergence radius, so does h^{-1} .

Proposition: If we choose $\mathbf{v}_k(k) = \mathbf{v}_1(1)^k$, then $h_k(x) = h_1(x)^k =: h(x)^k$.

Proof: Consider the ratio $r(x) = h_k(x)/h(x)^k$. It holds that $r(\phi(x)) = r(x)$ because

$$r(\phi(x)) = h_k(\phi(x))/h(\phi(x))^k = \frac{\lambda^k h_k(x)}{\lambda^k h_1(x)^k} = r(x).$$

But then all the k -derivatives $r^{(k)}(0)$, $k \geq 1$, of r at $x = 0$ vanish because $\phi(0) = 0$ and $\phi'(0) = \lambda \neq 1$. Therefore r is constant on $x \in I_c$. Choosing $\mathbf{v}_k(k) = \mathbf{v}_1(1)^k$, then $r(0) = \lim_{x \rightarrow 0} h_k(x)/h(x)^k = \mathbf{v}_k(k)/\mathbf{v}_1(1)^k = 1$, so $r(x) = 1$. One possible way to achieve this is to assume without loss of generality that $\mathbf{v}_1(1) = h_1 = 1 = \mathbf{v}_k(k)$.

Corollary: If $\mathbf{v}_k(k) = \mathbf{v}_1(1)^k$, the upper-triangular matrices V and V^{-1} in (6) are the Carleman transfer matrices of the power series h and h^{-1} in (8); similarly, the matrix D_λ is the transfer matrix of the linear power series $x \rightarrow \lambda x$.

Proof: by definition $V(k, k') = \mathbf{v}_k(k') = \begin{bmatrix} x^{k'} \end{bmatrix} h_k(x) = \begin{bmatrix} x^{k'} \end{bmatrix} h(x)^k$ and by Lagrange inversion formula, $V^{-1}(k, k') = \begin{bmatrix} x^{k'} \end{bmatrix} h^{-1}(x)^k$.

2.2. The critical case $\lambda = \alpha_1 = 1$ and Jordanization of P . Let us consider the critical case, $\alpha_0 = 0$ and $\alpha_1 = 1$. The transfer matrix P has only 1s on its diagonal, and is subsequently not (infinite-)diagonalizable. Let us search instead for a family of column vectors $\mathbf{w}_1, \mathbf{w}_2, \dots$, such that

$$P\mathbf{w}_1 = \mathbf{w}_1$$

and, if $k > 1$,

$$P\mathbf{w}_k = \mathbf{w}_{k-1} + \mathbf{w}_k,$$

which corresponds to the Jordanization of the infinite matrix P . A solution is given by $\mathbf{w}'_1 = (1, 0, \dots)$, $\mathbf{w}_k(k') = 0$ if $k' > k$, $\mathbf{w}_k(1) = 0$ if⁵ $k > 1$ and otherwise

$$\mathbf{w}_k(k') = \frac{1}{P(k'-1, k')} \left(\mathbf{w}_{k-1}(k'-1) - \sum_{i=k'+1}^k P(k'-1, i) \mathbf{w}_k(i) \right)$$

With W the matrix whose columns are the \mathbf{w}_k , I the infinite identity matrix and S the infinite (shift) matrix with 1s on its superdiagonal, we then have

$$PW = W(I + S)$$

and the powers of P satisfy

$$P^n W = W \sum_{k=0}^n \binom{n}{k} S^k$$

Note that since $P(k, k+1) = k\alpha_2$ in the critical case, we have that

$$W(k, k) = \frac{\alpha_2^{-(k-1)}}{(k-1)!},$$

⁵This condition is not necessary, but it simplifies the computation.

and because $W(k, k) \neq \lambda^k$ for some λ , W cannot be the transfer matrix of some power series. By construction, the first row of W is $(1, 0, \dots)$.

Define $\sigma(k, k') = 0$ if $k' \leq k$ together with

$$\sigma(k, k') = \sum_{1=i_1 < i_2 < \dots < i_{k+1}=k'} \prod_{l=1}^k P(i_l, i_{l+1}) \text{ if } k' > k \geq 1$$

and, if $k = 0$, $\sigma(k, k') = \delta_{1, k'}$. For example, $\sigma(2, 4) = P(1, 2)P(2, 4) + P(1, 3)P(3, 4)$, $\sigma(2, 3) = P(1, 2)P(2, 3)$. Then, we conjecture that

$$W^{-1}(k, k') = \sigma(k - 1, k'), \quad k' \geq k \geq 1.$$

We get:

Proposition: Since we have $x_n = \mathbf{e}'_1 P^n \mathbf{x}$, observing $\mathbf{e}'_1 W = \mathbf{e}'_1 W^{-1} = \mathbf{e}'_1$ and $\mathbf{e}'_1 S^k = \mathbf{e}'_{k+1}$,

$$(10) \quad x_n = \left(\sum_{k=0}^n \binom{n}{k} \mathbf{e}'_{k+1} \right) W^{-1} \mathbf{x} = x + \sum_{k=1}^n \binom{n}{k} \sum_{k' \geq k+1} \sigma(k, k') x^{k'}.$$

This constitutes the infinite-dimensional matrix representation of x_n in the critical case.

Slow extinction in the critical case. With $\phi(0) = 0$ and $\phi'(0) = \alpha_1 = 1$, consider the critical dynamics,

$$x_{n+1}(x) = \phi(x_n(x)), \quad x_0(x) = x.$$

Suppose $x_n(x)$ approaches 0 as n gets large. An order-two Taylor development of ϕ near $x = 0$ therefore gives

$$\begin{aligned} x_{n+1}(x) &\approx \phi'(0) x_n(x) + \frac{1}{2} \phi''(0) x_n(x)^2 \\ &= x_n(x) + \frac{1}{2} \phi''(0) x_n(x)^2, \end{aligned}$$

leading (assuming $\alpha_2 := \phi''(0) < 0$) to

$$(11) \quad x_n(x) \sim x / (1 - (n\alpha_2)/2), \text{ as } n \text{ is large,}$$

indeed going to 0. We conclude:

Proposition: For a critical dynamics (1) for which $\alpha_1 = 1$, if $\alpha_2 := \phi''(0) < 0$, $n x_n(x) \rightarrow 2x / (-\alpha_2)$, and x_n vanishes at slow algebraic rate n^{-1} .

2.3. Special models arising in population dynamics. Because we are interested in population dynamics systems for which $x_n \geq 0$ is the size of some population at time n , we shall limit ourselves to dynamical systems of type (1) generated by ϕ with $\phi(0) = 0$ and $\alpha_1 = \phi'(0) > 0$, $\alpha_1 \neq 1$ (recall however that the latter construction leading to (6) and (8) holds even if $\alpha_1 < 0$). The initial condition x will be assumed to belong to the domain $[0, x_b]$ where $x_b = \inf(x > 0 : \phi(x) = 0) \leq x_c^+$, possibly with $x_b = \infty$ if $x_c^+ = \infty$. We also need to assume that the maximal value ϕ^* that the function ϕ can take on $[0, x_b]$ is $\leq x_b$ so that ϕ maps $I = [0, x_b]$ onto $J \subseteq [0, x_b]$. If these conditions hold, the dynamics (1) will be said a population

dynamics model.

Examples of population models:

(i) (logistic map) $\phi(x) = rx(b-x)$ with $4 \geq \alpha_1 = br > 0$, $\alpha_2 = -r < 0$, $x_b = b > 0$, $x_c^+ = \infty$. Here $I = [0, b]$ and $J = [0, rb^2/4]$.

(ii) (homographic map) $\phi(x) = rx/(1+ax)$ with $\alpha_1 = r > 0$, $\alpha_2 = -ar < 0$, $x_b = x_c^+ = \infty$. Here $I = [0, \infty]$ and $J = [0, r/a]$.

(iii) (Ricker map) $\phi(x) = rx \exp(-ax)$ with $\alpha_1 = r > 0$, $\alpha_2 = -ar < 0$, $x_b = x_c^+ = \infty$. Here $I = [0, \infty]$ and $J = [0, \phi^* = 1/a] \subset I$.

(iv) $\phi(x) = x_* - \sqrt{a(x_* - x)^2 + b}$ with $a, b, x_* > 0$, $ax_*^2 + b = x_*^2$. Here $I = [0, 2\sqrt{b/(1-a)}]$ and $J = [0, \sqrt{b/(1-a)}(1 - \sqrt{1-a})]$. Note $\phi'(0) = \alpha_1 = a < 1$. For this map, $\phi(x) = q^{-1}(aq(x) + b)$ where $q(x) = (x - x_*)^2$.

Note that models (i) and (iii) are single-humped maps, meaning a typically smooth and non-negative function with exactly one critical point (where ϕ' vanishes), a maximum, and at most one point of inflection to the right of the maximum.

2.4. Invariant measure of specific maps. Consider a map ϕ from some interval I onto $J \subseteq I$. Then the invariant measure μ solves the Perron-Frobenius equation $\mu = \phi^{-1} \circ \mu$, [6] and [13]. When ϕ is chaotic, a density solution f possibly exists, thereby solving $f(x) = \int_I \delta(x - \phi(y)) f(y) dy$, [19]. This is also

$$(12) \quad f(x) = \sum_{y: \phi(y)=x} \frac{f(y)}{|\phi'(y)|} = \sum_{k=1}^{K(x)} f(\phi_k^{-1}(x)) |\phi_k^{-1}(x)'| = \sum_{k=1}^{K(x)} \frac{f(\phi_k^{-1}(x))}{|\phi'(\phi_k^{-1}(x))|},$$

where $K(x)$ is the number of ϕ -antecedents of x belonging to the set $\{x : f(x) > 0\}$.

Consider now a non-monotone map ϕ from some interval I onto itself (a surjection whose domain and range coincide with I). For almost all initial condition x (drawn at random uniformly on $[0, 1]$), for almost all $\phi'(0)$ in a chaotic range of ϕ , a density solution f of (12) can exist and be made explicit. For instance, consider the case investigated by [14] where in addition: $\forall x \in I$, $\phi^{-1}(x)$ is made of K antecedents (independently of x).

Given ϕ and $\kappa \in \mathbb{R}$, consider then the Schröder functional equation, with unknown function s ,

$$s(\phi(x)) = \kappa s(x).$$

Taking the derivative and then the modulus, $|\phi'(x)| |s'(\phi(x))| = |\kappa| |s'(x)|$. Thus for each branch k , $|\phi'(\phi_k^{-1}(x))| |s'(x)| = |\kappa| |s'(\phi_k^{-1}(x))|$, showing from (12), see [14], that if $|\kappa| = K$

$$f(x) = |s'(x)|.$$

This shows that for chaotic maps ϕ mapping some interval I into itself and for which $\forall x \in I$, $\phi^{-1}(x)$ is made of K antecedents (independent of x), if it has an invariant density, it can be explicitly obtained from the solution of a Schröder functional equation. But s can also be related to h , the Carleman function of ϕ .

Corollary: Let ϕ map I onto itself with $K(x) = K$ and $\lambda = \phi'(0) > 0$. Recalling the representation (8) of ϕ : $h(\phi(x)) = \lambda h(x)$, with $\alpha = \log_K \lambda$, we get $s(x) = h(x)^{1/\alpha}$ and so $f(x) = \frac{1}{|\alpha|} |h(x)^{1/\alpha-1} h'(x)|$.

This shows that in some cases, (8) can also be useful for the computation of the invariant density of ϕ . Let us illustrate these facts for the logistic map:

- With $I = [0, 1]$, consider the logistic map $\phi(x) = rx(1-x)$ with $b = 1$ and $r = 2$. For this map, $K = 2$ but ϕ maps $[0, 1]$ to $[0, 1/2] \subset [0, 1]$. Then, for every $x \in (0, 1/2)$, $\phi(x) = h^{-1}(2h(x))$ where $h(x) = -\log(1-2x)/2$ and $h^{-1}(x) = (1 - e^{-2x})/2$, the inverse of h . The invariant measure is a Dirac at the fixed point $x = 1/2$ of ϕ . If $2 < r < r \simeq 3.5699456$ (the Feigenbaum constant), the invariant measure is a Dirac measure concentrating at the fixed points obtained in the period doubling process of the logistic map.

- Consider the logistic map $\phi(x) = rx(1-x)$ with $b = 1$ and $r = 4$: ϕ maps $[0, 1]$ into $[0, 1]$ and $K = 2$ for all x . Then $\phi(x) = h^{-1}(4h(x))$ where $h(x) = \arcsin(\sqrt{x})^2$ and $h^{-1}(x) = \sin(\sqrt{x})^2$. The invariant measure is $\mu(dx) = f(x)dx$ where $f(x) = \pi^{-1}(x(1-x))^{-1/2} = |h^{1/2}(x)'| = |s'(x)|$ where $s(x) = \arcsin(\sqrt{x})$ solves $s(\phi(x)) = 2s(x)$.

We now wish to further investigate some aspects of the invariant density problem for the general logistic map.

3. INVARIANT DENSITIES FOR $\phi(x) = rx(1-x)$

Let us further consider the logistic map $\phi(x) = rx(1-x)$ with $b = 1$ and assume now $4 > r = \lambda > 2$. This map is not into I but onto only. In this case the dynamics moves to the interval $[\phi(r/4), r/4]$ in finite time and stays there⁷: the support of the invariant measure is expected to lie within this interval and, whenever $4 \geq r > 3.5699456$ (the Feigenbaum constant for the onset of chaos), the set of points with positive measure possibly consists in a finite union of disconnected subintervals embedded within this support. Within the interval $r \in [3.5699456..., 4]$, it is widely known that there are subintervals where the map ϕ is not chaotic (has negative Lyapounov exponent) corresponding to x_n reaching a cycle of any length. In these subintervals, the invariant measure is expected to be a Dirac measure.

The purpose of this section is to exhibit some features of the invariant density in the chaotic regimes, when this density exists. In the following, we will use the notation $\alpha_n(r) := \phi_n(1/2)$.

To illustrate this section, we will often refer to a specific value of r , $r \simeq 3.6785735$, which is the one studied by Ruelle in [16]. This value is the smallest r for which there is a density with full support $[1/r, r/4]$, and it solves $(r-2)(r^2-4) = 16$.

⁶In this case, the inverse function of $h(x) = \arcsin(\sqrt{x})^2$ is $\sin(\sqrt{x})^2$, restricted to the interval $[0, \pi/2]$ whereas its analytic continuation h^{-1} is the same function now defined on the whole positive real line (its maximal domain of convergence).

⁷ $x > r/4$ has no antecedent and cannot be reached, the function is strictly increasing on $[0, \phi(r/4)]$, and the interval $[\phi(r/4), r/4]$ is mapped onto itself.

3.1. Divergence points of the invariant density. When $r = 4$, the invariant density diverges like $x^{-1/2}$ near 0 and $(1-x)^{-1/2}$ near 1, $\{0, 1\}$ being the boundary of the support of the invariant density. When $r = 4$, ϕ maps $1/2$ to 1 and then to 0, a fixed point of the dynamics: $\alpha_0(r) = 1/2$, $\alpha_1(r) = 1$ and $\alpha_2(r) = 0$. The density diverges at $\{0, 1\}$, the successive image of $1/2$ under ϕ .

When $r \simeq 3.6785735$, ϕ maps $1/2$ to $r/4$ and then to $1/r$ and finally to $1 - 1/r$, a fixed point of the dynamics: $\alpha_0(r) = 1/2$, $\alpha_1(r) = r/4$, $\alpha_2(r) = 1/r$ and $\alpha_3(r) = 1 - 1/r$. The density is expected to diverge at points $\{1/r, 1 - 1/r, r/4\}$, the successive images of $1/2$ under ϕ .

This seems to be a generic fact.

For a generic value of r , when $x = \alpha_1(r) = r/4$, there is only one antecedent, namely $y = 1/2$ and there, $\phi'(y = 1/2) = 0$, so, in view of (12), the invariant density $f(x)$ of the mass at $x = r/4$ must diverge, if it exists. This is also true for $x = \phi_1(r/4) = \phi_2(1/2) = \alpha_2(r)$: for this x indeed, $y = r/4$ is an antecedent and because $f(y)$ diverges for this value of y , so does $f(x)$. By induction,

Proposition: For all $x = \alpha_n(r)$, $n \geq 1$, there is a divergence of the invariant density.

For the Ruelle value of r , the sequence $\phi_n(1/2) = \alpha_n(r)$, $n \geq 1$, stops (converges) after three steps, so one expects a divergence of the density f in that point which is $1 - 1/r$ (the fixed point of ϕ) but also at the intermediate points $\alpha_1(r) = r/4$, $\alpha_2(r) = 1/r$. Following this argument, when $r \simeq 3.89087$ (defined by $\alpha_5(r) = \alpha_3(r) \neq \alpha_4(r)$), one expects a divergence of the density at the four points $\alpha_1(r) = r/4 = \phi_1(1/2)$, $\alpha_2(r) = \phi_2(1/2)$, $\alpha_3(r) = \phi_3(1/2)$ and $\alpha_4(r) = \phi_4(1/2)$. The sequence stops there because for this value of r , $\alpha_5(r) = \alpha_3(r)$, $\alpha_6(r) = \alpha_4(r)$ and no new $\alpha_n(r)$ is expected, this sequence oscillating between $\alpha_3(r)$ and $\alpha_4(r)$ with $\alpha_3(r) > \alpha_4(r)$. This concerns special values of r for which f has full support and finitely many peaks.

This poses the following general question: let \mathcal{C}_m be a cycle of length m of ϕ . Consider the set

$$\mathcal{R} := \{r \text{ in the chaotic regime: } \exists n, m < \infty: \alpha_n(r) \in \mathcal{C}_m \text{ and } \alpha_{n-1}(r) \notin \mathcal{C}_m\}.$$

If $r \in \mathcal{R}$ and if corresponding map ϕ admits a density f , then there must be $n + m - 1$ peaks where f diverges, by the arguments above. If $r \notin \mathcal{R}$ and if f exists, then f has infinitely many peaks within its support. We don't know if \mathcal{R} is dense within the subset of values of r in the chaotic regime, nor do we have a specific value of $r \notin \mathcal{R}$. For an $r \notin \mathcal{R}$, the sequence $\{\alpha_n(r)\}$ does not enter a cycle in finite time and one expects a density diverging on this countable set values for x within $[\phi(r/4), r/4]$.

It remains to determine the type of the divergence of f . We shall develop our arguments for the Ruelle value of r . We have

$$\phi(y) = \phi\left(\frac{1}{2} + \left(y - \frac{1}{2}\right)\right) = \frac{r}{4} - r\left(y - \frac{1}{2}\right)^2, \quad \phi'(y) = r(1 - 2y).$$

Put $x = r/4 - \varepsilon$ and $y = 1/2 - \varepsilon'$ with $\varepsilon, \varepsilon' > 0$. Because $\phi'(1/2) = 0$, we have $|\phi'(y)| = 2r\varepsilon'$ and $\phi(y) = r/4 - r\varepsilon'^2 = x = r/4 - \varepsilon$. Thus $r\varepsilon'^2 = \varepsilon$.

In view of

$$f(x) = \sum_{y:\phi(y)=x} \frac{f(y)}{|\phi'(y)|}, \text{ therefore } f(r/4 - \varepsilon) \sim \frac{f(1/2)}{2\sqrt{r\varepsilon}}.$$

Similarly, with some linear relation between ε , ε' and ε'' , the antecedents of $1/r + \varepsilon$ are $r/4 - \varepsilon'$ and $1 - r/4 + \varepsilon''$, the latter being outside of the support of f so its value there is zero. The antecedents of $1 - 1/r + \varepsilon$ are $1/r + \varepsilon'$ and $1 - 1/r - \varepsilon''$. The antecedents of $1 - 1/r - \varepsilon$ are $1 - 1/r + \varepsilon'$ and $1/r - \varepsilon''$, the latter being outside of the support of f . With these relations, we find that

$$(1/r + \varepsilon) \sim a/\sqrt{\varepsilon}, \quad f(1 - 1/r - \varepsilon) \sim b/\sqrt{\varepsilon}, \quad f(1 - 1/r + \varepsilon) \sim c/\sqrt{\varepsilon}$$

with

$$a := \frac{f(1/2)}{|\phi'(1/r)| |\phi'(1/r)|^{1/2}}, \quad b := \frac{a |\phi'(1/r)|}{|\phi'(1/r)| - 1}, \quad c := \frac{a |\phi'(1/r)|^{3/2}}{|\phi'(1/r)| - 1}.$$

Proposition: There is an algebraic divergence of f at the peaks of order $-1/2$ whereby f is integrable.

3.2. Disconnected invariant measure support. As a polynomial in the variable r , $\alpha_n(r)$ has degree $2^n - 1$. When r is larger than the Ruelle value $r \simeq 3.6785735$, whenever the regime is chaotic, the support of the invariant measure is made of one single piece, namely the full interval $[\alpha_2(r) = \phi(r/4), \alpha_1(r) = r/4]$. The above value of r is when $\alpha_3(r)$ is a fixed point of ϕ , namely $\alpha_3(r) = \phi(\alpha_3(r)) = \alpha_4(r)$. Because $\alpha_3(r) = 1 - 1/r$ and $\phi(1/r) = 1 - 1/r$, this Ruelle value of r is also obtained when $\alpha_2(r) = r^2(1 - r/4)/4 = 1/r$, indeed leading to $r \simeq 3.6785735$. When r becomes slightly less than the Ruelle value, the support of the invariant density of ϕ splits in the two pieces $[\alpha_2(r), \alpha_4(r)] \cup [\alpha_3(r), \alpha_1(r)]$, because $\alpha_4(r) < \alpha_3(r)$. These two pieces each split again in two additional pieces for the value of r for which $\alpha_5(r) = \alpha_7(r)$ and $\alpha_6(r) = \alpha_8(r)$, corresponding to $\alpha_5(r)$ and $\alpha_6(r)$ being respectively one of the two known fixed points of ϕ_2 (which are not the fixed points $\{0, 1 - 1/r\}$ of $\phi_1 = \phi$). Slightly below this value of r , the support of the invariant density of ϕ splits into the four pieces $[\alpha_2(r), \alpha_8(r)] \cup [\alpha_6(r), \alpha_4(r)] \cup [\alpha_3(r), \alpha_7(r)] \cup [\alpha_5(r), \alpha_1(r)]$: this transition is seen to occur at $r \simeq 3.5925722$. The next third step generates the $2^3 = 8$ pieces

$$\begin{aligned} & [\alpha_2(r), \alpha_{16}(r)] \cup [\alpha_{12}(r), \alpha_8(r)] \cup [\alpha_6(r), \alpha_{14}(r)] \cup [\alpha_{10}(r), \alpha_4(r)] \cup \\ & [\alpha_3(r), \alpha_{15}(r)] \cup [\alpha_{11}(r), \alpha_7(r)] \cup [\alpha_5(r), \alpha_{13}(r)] \cup [\alpha_9(r), \alpha_1(r)] \end{aligned}$$

whenever $\alpha_{16}(r) = \alpha_{12}(r)$, $\alpha_{14}(r) = \alpha_{10}(r)$, $\alpha_{15}(r) = \alpha_{11}(r)$ and $\alpha_{13}(r) = \alpha_9(r)$ corresponding to $\alpha_{12}(r)$, $\alpha_{10}(r)$, $\alpha_{11}(r)$ and $\alpha_9(r)$ being the four fixed points of ϕ_4 which are neither the ones of ϕ_1 nor the ones of ϕ_2 . For some special value of r which can be computed, the four pieces split into these eight pieces.

This binary splitting process of the support can be iterated until one reaches the critical Feigenbaum value of r , namely $r \simeq 3.5699456$ where the invariant measure is expected to be singular (whose support consists in an uncountable number of points). For values of r less than this Feigenbaum value and larger than three, the invariant measure is a Dirac measure concentrated on the period two cyclic points

appearing in the period doubling process onsetting after $r = 3$.

3.3. Invariant measure for some specific values of r . Let us consider the Ruelle value of r . For this value of r , the quartic map

$$\psi(x) = (r-2)^2 x(1-x)[1 + (r-2)x(1-x)]$$

maps $[0, 1]$ into $[0, 1]$ and it has $K(x) = 2$ branches for each $x \in [0, 1] \setminus \{1/2\}$. Furthermore, $\psi(x) = q_2(x) := q \circ q(x)$ where $q(x) = (r-2)x(x-1)$ and, with $a(x) = (r-1-rx)/(r-2)$

$$\phi(x) = a^{-1}(q(a(x))) \text{ and } \phi_2(x) := \phi \circ \phi(x) = a^{-1}(\psi(a(x))).$$

As stated before, this value is also one for which $\alpha_3(r) = \alpha_4(r)$ ($\alpha_3(r)$ is a fixed point of ϕ).

Let now h be the Carleman function associated to ψ , defined by $\psi(x) = h^{-1}(\lambda h(x))$, $\lambda = \psi'(0) = (r-2)^2 > 2$, as from (8). The dynamical system generated by ψ has an absolutely continuous invariant measure, given by $f_\psi(x) = |s'(x)|$ where $s(\psi(x)) = 2s(x)$ and $s(x) = h^{1/\alpha}(x)$, $\alpha = \log_2 \lambda > 1$. Therefore so does the dynamical system generated by ϕ_2 with invariant density given by $f_{\phi_2}(x) = |a'(x)| f_\psi(a(x))$, with support $[1/r, 1-1/r]$. And then so does the dynamical system generated by ϕ itself, with the density $f_\phi(x) = \frac{1}{2}(f_{\phi_2}(x) + \phi^{-1} \circ f_{\phi_2}(x))$, where $\phi^{-1} \circ f_{\phi_2}(x) = |\phi'(x)| f_{\phi_2}(\phi(x))$ is the image density of f_{ϕ_2} under ϕ^{-1} . The support of f_ϕ is $[1/r, r/4] = [1/r, 1-1/r] \cup \phi^{-1}[1/r, 1-1/r]$. We conclude that there is a Carleman inspired analytic expression of the invariant density for the logistic model when r takes on the Ruelle value. This is not the only one case.

Looking for an affine function $a(x) = c - dx$ such that $\phi_4(x) = a^{-1}(\Psi(a(x)))$ for some map $\Psi = \psi \circ \psi$ with $\psi(x) = ax(1-x)[1 + bx(1-x)]$, such that Ψ maps $[0, 1]$ onto itself, leads to another solution to the above Ruelle scenario, namely: $a \simeq -1.7214305$, $b \simeq 0.1592945$, $c \simeq -2.2648957$, $d \simeq -5.5297915$ and $r \simeq 3.5925722$. The latter is the value of r where the support of the invariant measure splits from 2 pieces to 4. In that case, we have $\lambda := \Psi'(0) = a^2$, Ψ having two branches on $[0, 1] \setminus \{1/2\}$. Using the same Carleman matrix approach as previously stated, it is possible to compute the invariant measure associated to the mapping Ψ (now with h the Carleman function of Ψ and $s(x) = h^{1/\alpha}(x)$ for $\alpha = \log_2 a^2$). From that result and following the arguments developed for the Ruelle value, one can clearly compute the invariant density first of the function ϕ_4 on a given subset of its support, namely f_{ϕ_4} , and then get the invariant density of ϕ on its support by

$$f_\phi = \frac{1}{4}(f_{\phi_4} + \phi^{-1} \circ f_{\phi_4} + \phi^{-2} \circ f_{\phi_4} + \phi^{-3} \circ f_{\phi_4}).$$

Proposition: For the particular value of $r \simeq 3.5925722$, the invariant density of ϕ can be computed.

Although for $r \simeq 3.6785735$ and for $r \simeq 3.5925722$, the maps ϕ were not from $[0, 1]$ onto $[0, 1]$, the powers ϕ_2 and ϕ_4 of these maps were shown to be affine-conjugate to maps ψ and Ψ which now map $[0, 1]$ onto $[0, 1]$, thereby amenable to the Schröder

and Carleman representation of the invariant densities.

4. THE GENERAL CASE $\alpha_0 \neq 0$

Let $\phi_0(x) = \sum_{k \geq 0} \alpha_k x^k = \alpha_0 + \phi(x)$, now with $\alpha_0 = \phi_0(0) \neq 0$. Consider now the dynamical system

$$(13) \quad x_{n+1} = \phi_0(x_n), \quad x_0 = x.$$

Let

$$(14) \quad P_0(k, k') = \left[x^{k'} \right] \phi_0(x)^k, \quad k, k' \geq 0.$$

Note P_0 no longer is triangular. Specifically, for $k, k' \geq 1$

$$\begin{aligned} P_0(k, k') &= \left[x^{k'} \right] (\alpha_0 + \phi(x))^k = \sum_{l=0}^k \binom{k}{l} \alpha_0^{k-l} \left[x^{k'} \right] \phi(x)^l \\ &= \sum_{l=1}^k \binom{k}{l} \alpha_0^{k-l} P(l, k') = \sum_{l=1}^k \binom{k}{l} \alpha_0^{k-l} \widehat{B}_{k', l}(\alpha_1, \alpha_2, \dots, \alpha_{k'-l+1}), \end{aligned}$$

together with $P_0(0, k') = \delta_{0, k'}$, $P_0(k, 0) = \alpha_0^k$, for $k \geq 0$.

$$\text{Hence } P_0 = \overline{B} \overline{P} \text{ where } \overline{B}(k, l) = \binom{k}{l} \alpha_0^{k-l}, \quad k \geq l \geq 0 \text{ and } \overline{P} = \begin{bmatrix} 1 & 0 & \cdots & \cdots \\ 0 & & & \\ \vdots & P & & \\ \vdots & & & \end{bmatrix},$$

$$\text{so with } P_0 = \begin{bmatrix} 1 & 0 & \cdots & \cdots \\ \alpha_0 & & & \\ \alpha_0^2 & BP & & \\ \vdots & & & \end{bmatrix} \text{ and } B(k, l) = \binom{k}{l} \alpha_0^{k-l}, \quad k \geq l \geq 1, \text{ the incom-}$$

plete lower-triangular Pascal (binomial) matrix. Note $(\overline{B}\mathbf{x})_k = (\alpha_0 + x)^k$: the full Pascal matrix \overline{B} is the Carleman matrix of the shift function $b(x) = \alpha_0 + x$. It also holds $\overline{B}(k, k) = 1$, $k \geq 0$ and P_0 admits a LU factorization.

Then, with $\mathbf{e}'_1 = (0, 1, 0, 0, \dots)$ and $\overline{\mathbf{x}}' = (1, x, x^2, \dots)$,

$$(15) \quad x_n = \mathbf{e}'_1 P_0^n \overline{\mathbf{x}}.$$

This shows that a general non-linear dynamical system (13) generated by ϕ_0 with $\phi_0(0) \neq 0$ is also a linear infinite-dimensional system with complete ‘transfer matrix’ P_0 . Because P_0 no longer is triangular, its eigenvalues are not known nor is its diagonalization easy, if even possible. So P_0^n is very complex. Note in particular that $x_n(x=0) = \mathbf{e}'_1 P_0^n \mathbf{e}_0 = P_0^n(1, 0)$, the $(1, 0)$ -entry of P_0^n .

In some cases however, the general problem (13) can be taken back to a simpler problem of type (1), see [9].

Let $y_n = x_n - \rho$ for some real number ρ . Then

$$y_{n+1} = \phi_0(y_n + \rho) - \rho, \quad y_0 = x - \rho.$$

Suppose there is a real number ρ such that $\phi_0(\rho) = \rho$ (the map ϕ_0 has a fixed point). Then

$$(16) \quad y_{n+1} = \bar{\phi}(y_n) := \phi_0(y_n + \rho) - \rho, \quad y_0 = x - \rho,$$

where $\bar{\phi}(0) = \phi_0(\rho) - \rho = 0$. If this is the case, the new system (16) now generated by $\bar{\phi}$ is of the form (1). It can be solved as a (1) model with the new

$$\bar{\phi}(y) = \sum_{k \geq 0} \alpha_k (y + \rho)^k - \rho =: \sum_{l \geq 1} y^l \bar{\alpha}_l,$$

where $l! \bar{\alpha}_l = \sum_{k \geq l} \frac{\rho^{k-l}}{(k-l)!} (k! \alpha_k)$ is of convolution type. And then, assuming $\bar{\lambda} := \bar{\alpha}_1 \neq 1$,

$$(17) \quad x_n = \bar{h}^{-1} \left(\bar{\lambda}^n \bar{h}(x - \rho) \right) + \rho,$$

where \bar{h} is associated to the new generator $\bar{\phi}$ (just like h in (8) was to ϕ in (1)). Introducing $\tilde{h}(x) = \bar{h}(x - \rho)$, this is also $x_n = \tilde{h}^{-1} \left(\bar{\lambda}^n \tilde{h}(x) \right)$ similar to (8), except that $\tilde{h}(0) \neq 0$. Note also that $\bar{\phi}'(0) =: \bar{\alpha}_1 = \phi_0'(\rho)$ is not necessarily > 0 . And depending on $|\phi_0'(\rho)| < 1$ (or > 1), ρ is a stable (unstable) fixed point of (13). Equivalently, (see e.g. [9])

Proposition: With Q the upper-triangular Carleman matrix of $\bar{\phi}$ (easily diagonalizable with $VQ = D_{\bar{\lambda}}V$ where $D_{\bar{\lambda}} = \text{diag}(1, \bar{\lambda}, \bar{\lambda}^2, \dots)$), the Carleman matrix P_0 of ϕ_0 therefore obeys

$$(18) \quad P_0 = B_\rho^{-1} Q B_\rho = (V B_\rho)^{-1} D_{\bar{\lambda}} V B_\rho,$$

where B_ρ is the lower-triangular Carleman matrix of the shift function $b_\rho(x) = x - \rho$: $B_\rho(k, l) = (-1)^{k-l} \binom{k}{l} \rho^{k-l}$ and $B_\rho^{-1}(k, l) = \binom{k}{l} \rho^{k-l}$.

Remarks:

- The only Carleman matrices which are lower-triangular are the ones associated to an affine map as the one above.
- With $\phi_0(\rho)' = (1, \phi_0(\rho), \phi_0(\rho)^2, \dots)$ and $\boldsymbol{\rho}' = (1, \rho, \rho^2, \dots)$, for all fixed point ρ of ϕ_0 : $P_0 \boldsymbol{\rho}' = \phi_0(\rho)' = \boldsymbol{\rho}'$ showing that $\boldsymbol{\rho}'$ is a right eigenvector of P_0 associated to its eigenvalue 1.
- If the map ϕ_0 has more than one real fixed point, the latter construction holds for any of these fixed points, showing that (17) is not unique.
- If the map ϕ_0 has no real fixed point, P_0 is not real-diagonalizable.

For example, the matrix P_0 associated to the map $\phi_0(x) = x + \alpha_0/(x+1)$ with $\phi_0(0) = \alpha_0 \neq 0$ is not diagonalizable. However, this model has a fixed point at infinity. Exchanging ∞ and 0 can be done while using the transformation $y = 1/x$. The dynamics for the y s is thus $y_{n+1} = \phi(y_n)$, $y_0 = y = 1/x$, with

$$\phi(y) = 1/\phi_0(1/y) = \frac{y(1+y)}{1+y+\alpha_0 y^2},$$

now with a fixed point at $y = 0$ and of type (1), with ϕ rational. We have $\phi'(0) = \alpha_1 = 1$ and $\phi''(0) = \alpha_2 = 0$, a critical model, therefore amenable to Jordanization.

Another example is $\phi_0(x) = x + x^2 + \alpha_0$ with $\phi_0(0) = \alpha_0 \neq 0$. This map has no real fixed point but it has two complex fixed points $\rho = \pm i\sqrt{c}$. The matrix P_0 associated to this map ϕ_0 is not real-diagonalizable but it is complex-diagonalizable. In such cases, only (15) holds, but not (17) where \bar{h} is real-valued.

4.1. An equivalent conjugation representation of ϕ_0 having a fixed point.

Assuming $\bar{\lambda} := \bar{\alpha}_1 = \phi'_0(\rho) \neq 1$, with $\tilde{h}(x) = \bar{h}(x - \rho)$, we obtained

$$x_1 = \phi_0(x) = \bar{h}^{-1}(\bar{\lambda} \bar{h}(x - \rho)) + \rho = \tilde{h}^{-1}(\bar{\lambda} \tilde{h}(x)),$$

where \bar{h} is associated to the generator $\bar{\phi}$. Define $g(x) := \tilde{h}(x) - \tilde{h}(0) = \bar{h}(x - \rho) - \bar{h}(-\rho)$, now obeying $g(0) = 0$ and $g(\rho) = -\bar{h}(-\rho)$. Clearly then, with $\bar{\mu} := (\bar{\lambda} - 1)\bar{h}(-\rho) = (1 - \bar{\lambda})g(\rho)$,

$$(19) \quad x_1 = \phi_0(x) = g^{-1}(\bar{\lambda}g(x) + \bar{\mu}),$$

showing that ϕ_0 is g -conjugate to the affine function $\bar{\lambda}x + \bar{\mu}$, with $c = \phi_0(0) = g^{-1}(\bar{\mu}) > 0$. This can be iterated to give

$$(20) \quad x_n = g^{-1}(\bar{\lambda}_n g(x) + \bar{\mu}_n), \text{ with } \bar{\lambda}_n = \bar{\lambda}^n \text{ and } \bar{\mu}_n = \bar{\mu} \left(1 + \bar{\lambda} + \dots + \bar{\lambda}^{n-1}\right) = g(\rho) \left(1 - \bar{\lambda}^n\right).$$

If $|\bar{\lambda}| < 1$ (ρ is a stable fixed point of ϕ_0), $x_n \rightarrow \rho$.

4.2. Population models with immigration. In population dynamics systems for which $x_n \geq 0$, we shall limit ourselves to dynamical systems of type (13) generated by ϕ_0 with $\phi_0(0) = c > 0$. No need to require here anymore that $\phi'_0(0) = \phi'(0) = \alpha_1 > 0$. The initial condition x will be assumed to belong to the domain $[0, x_b]$ where $x_b = \inf(x > 0 : \phi_0(x) = 0)$, possibly with $x_b = \infty$. Here $\phi_0(0) > 0$ interprets as an immigration rate. We also need to assume that the maximal value ϕ_0^* that $\phi_0(x)$ can take on $[0, x_b]$ is $\leq x_b$ so that ϕ_0 maps $I = [0, x_b]$ onto $J \subseteq I$. These ϕ_0 are amenable to the formalism (15) and (19).

Examples of population models with immigration $\phi_0(0) = c > 0$:

(i) (logistic map) $\phi_0(x) = c + rx(b - x)$, $r > 0$, $x_b = (br + \sqrt{\Delta}) / (2r) > b > 0$, $\Delta = (br)^2 + 4rc$, $x_c^+ = \infty$. Here $I = [0, x_b]$ and $J = [0, c + rb^2/4] \subseteq I$ provided $\phi_0^* = c + rb^2/4 \leq x_b$.

(i') (logistic map') $\phi_0(x) = c + rx(b - x)$, $r < 0$, $b > 0$. In this setup, $\phi'_0(0) = \phi'(0) = \alpha_1 = rb < 0$

- If $\phi_0(b/2) = c + r(b/2)^2 > 0$, then $x_b = x_c^+ = \infty$ and $I = J = [0, \infty]$.

- If $\phi_0(b/2) = c + r(b/2)^2 \leq 0$, then $x_b < \infty$. Here $I = [0, x_b]$ and $J = [0, c] \subseteq I$ provided $c \leq x_b$.

(ii) (homographic map) $\phi_0(x) = c + rx/(1 + ax)$ with $x_b = \infty = x_c^+$. Here $I = [0, \infty]$ and $J = [c, c + r/a] \subset I$. For this map, g in (19) is readily seen to be an homographic function itself, using the matrix representation of an homography, matrix product translating into composition of homographic maps.

(iii) (Ricker map) $\phi_0(x) = c + rx \exp(-ax)$ with $x_b = x_c^+ = \infty$. Here $I = [0, \infty]$ and $J = [c, c + 1/a] \subset I$.

(iv) $\phi_0(x) = c + x_* - \sqrt{a(x_* - x)^2 + b}$. Here $I = [0, x_b]$ and $J = [0, c + x_* - \sqrt{b}] \subseteq I$.

In these 4 examples, there is a smallest $\rho > 0$ for which $\phi_0(\rho) = \rho$.

(v) Consider the quadratic population model with immigration $\phi_0(x) = x + x^2 + c$ with $\phi_0(0) = c > 0$. In this example, $\phi_0(x)$ has no real fixed point and clearly with $x_{n+1} = \phi_0(x_n)$, $x_0 = x$, $x_n \sim \lambda(x, c)^{2^n}$, with $\lambda(x, c) > 1$ depending on both x and c . x_n drifts to ∞ at doubly exponential speed with n .

(vi) Consider the quadratic population model with immigration $\phi_0(x) = x + c/(x+1)$ with $\phi_0(0) = c > 0$. In this example, $\phi_0(x)$ has no real fixed point either and clearly with $x_{n+1} = \phi_0(x_n)$, $x_0 = x$, $x_n \sim \sqrt{2cn} \rightarrow \infty$, whatever $x > 0$. x_n drifts algebraically slowly to ∞ .

4.3. Logistic population models with or without immigration. Consider the logistic population dynamical system without immigration

$$(21) \quad x_{n+1} = rx_n(1 - x_n), x_0 = x,$$

where $r \in (0, 4]$. Let $y_n = ax_n + b$, with $b = (1 - a)/2$. Then, if $a = r/R$

$$(22) \quad y_{n+1} = c + Ry_n(1 - y_n), y_0 = y,$$

where

$$(23) \quad c = \frac{(r - 1)^2 - (R - 1)^2}{4R}.$$

For the dynamics y_n to be a logistic population dynamical system with immigration, mapping some interval I onto $J \subseteq I$, elementary algebra shows that the pair (r, R) must lie in one of the shaded zone Z of Figure 1 below, where $|R| \leq 4$ and $r \in (0, 4]$.

The detailed explanation of Figure 1 is as follows: for any value of R and r , there exists a c such that the dynamics $rx(1 - x)$ and $Ry(1 - y) + c$ are related. For the latter to be an admissible population dynamics including immigration, it has to satisfy several conditions. First, the number c has to be positive (which is related to the lines $R = r$ and $R = 2 - r$ on the graph). With this condition, the dynamics in y has to belong to one of three classes: $R > 0$ (corresponding to the vertically dashed area), $R < 0$ and the dynamics never reaches zero (the diagonally dashed area), or $R < 0$ and the dynamics reaches zero (the horizontally dashed area). The domain of the first-kind dynamics obviously corresponds to the superior half-plane, while the domains for the second and the third one are separated by $R = r(2 - r)/2$. Another condition to be an admissible dynamics is that it has to map one finite (because the $rx(1 - x)$ is defined on $[0, 1]$) domain into itself. This condition is related to the lines $R = r(r - 2)(r^2 - 2r - 4)/8$ for the first-kind dynamics, $R = -r$ for the second-kind one and $r^2 - R^2 - 2r - 2R - 4 = 0$ for the third-kind one. Finally, to be consistent with the $rx(1 - x)$ dynamics, the initial condition in y has to be mapped in an initial condition in x that belongs to $[0, 1]$, which is in relation to the line $r = R$.

This means that, starting from a dynamical system of type (22) for some R chosen in the range $|R| \leq 4$, for any value of r intersecting the shaded zone with the horizontal line of equation $y = R$, there is a positive immigration rate c given by (22) such that the dynamics of the y s is mapped into the dynamics (21) using the reverse affine transformation $x_n = (y_n - b) / a$. We conclude:

Proposition: With $|R| \leq 4$, if $r \in (0, 4]$ is in the shaded zone Z of the graph and also in the chaotic range for (21), then the logistic population dynamical system (22) with immigration rate c given by (23) is chaotic.

Starting from a logistic model without immigration and with reproduction rate $R > 0$ (not) in the chaotic region, adding immigration c can lead to a (c, R) model (22) either (chaotic) or non chaotic, depending on the chosen value of r : immigration can either (destabilize) stabilize a (non) chaotic system, [18]. We also note that when R is negative, but not too negative, there is no r large enough to lie in the chaotic region of (21).

Corollary: With $|R| \leq 4$, suppose $r \in (0, 4]$ is in the shaded zone Z of the graph. If $x_n = h^{-1}(r^n h(x))$ is the conjugacy representation (8) of x_n , then, with $a = r/R$ and $b = (1 - a) / 2$

$$y_n = ah^{-1}(r^n h((y - b) / a)) + b$$

is a conjugacy solution y_n of (22), started at y .

Corollary: If (22) is chaotic and corresponding (21) has an invariant density f , then (22) has the image invariant density: $\tilde{f}(y) = f((y - b) / a) / a$.

Example: Let $R = -4$ and $r = 4$ (in Z), so that $c = 1$, corresponding to the logistic population dynamical system with immigration

$$(24) \quad y_{n+1} = 1 - 4y_n(1 - y_n), y_0 = y.$$

Here $a = -1$ and $b = 1$. With $h(x) = \arcsin(\sqrt{x})^2$, we have

$$y_n = 1 - h^{-1}(4^n h((1 - y))),$$

and this dynamics is chaotic.

It can be checked that the representation (20) takes the alternative form

$$y_n = \sin^2 \left((-2)^n \arcsin(\sqrt{x}) + \frac{\pi}{6} (1 - (-2)^n) \right),$$

corresponding to $g(x) = \arcsin(\sqrt{x})$, $\bar{\lambda} = -2$ and $\rho = 1/4$ (with $g(\rho) = \pi/6$), in the notations of subsection 4.1.

(21) with $r = 4$ has the invariant density $f(x) = \pi^{-1} (x(1 - x))^{-1/2}$ and the logistic population dynamical system with immigration (24) has the invariant density $\tilde{f}(y) = f((y - 1) / (-1)) = \pi^{-1} (y(1 - y))^{-1/2}$, so identical to f (with Lyapounov exponent $\log 2$).

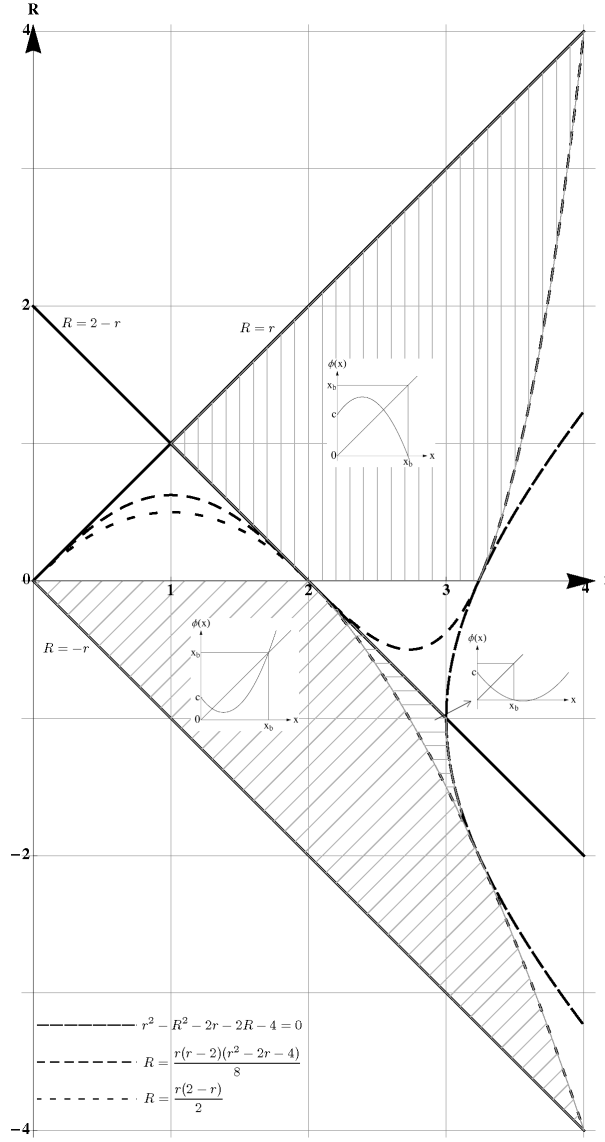


FIGURE 1. Map of the admissible correspondences between R and r . A point in the dashed areas corresponds to an admissible transformation. The miniatures of graphs indicate the kind of dynamics for $Ry(1-y) + c$.

5. CONCLUSION

Using Carleman linearization techniques applied to 1-dimensional discrete-time population dynamical systems, we gave a technique to compute the current population state x_n , for any initial point x , without actually computing the intermediate values x_1, \dots, x_{n-1} . This technique was shown to be related to the characterization of the invariant density measure, when it exists. But this is at the expense of the

computation of h and h^{-1} , which are “simple” special functions only in some exceptional situations, such as for specific parameter values of the logistic map with or without immigration. What “simple” means and the class of models for which these functions are “simple” and/or lead to chaotic behavior are largely open problems. This methodology has also recently proved useful in the context of discrete-time branching process for which the map ϕ is absolutely monotone: for the family of so-called generalized linear-fractional branching processes first introduced in [17] and further studied in [8], the function h has a simple structure and the iteration of ϕ does not lead of course to chaos being one-to-one on the unit interval.

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